MORE COMMUTATOR INEQUALITIES FOR HILBERT SPACE OPERATORS

Wasim Audeh
Department of Basic Sciences
Petra University
Amman, Jordan
e-mail: waudeh@uop.edu.jo

Abstract

We present general singular value inequalities for \( n \)th order Audeh generalized commutator from them recent results for commutators due to Bhatia-Kittaneh, Kittaneh, Hirzallah-Kittaneh, Hirzallah, and Wang-Due are special cases. Several applications are given.

1. Introduction

Let \( B(H) \) denote the space of bounded linear operators on a complex separable Hilbert space \( H \), and let \( K(H) \) denote the two-sided ideal of compact operators in \( B(H) \). An operator of the form \( AX -XA \) is called a commutator, and an operator of the form \( AX -XB \) is called a generalized commutator. Various singular value inequalities for the commutator or the generalized commutator are obtained by different authors. In this paper, the author uses the \( n \)th order Audeh generalized commutator \( A_1X_1 + \cdots + A_mX_m - Y_1B_1 - \cdots - Y_nB_n \) to generalize the commutator and consider analogous singular value inequalities.
Kittaneh has proved in [7] that if $A, B, X, Y \in B(H)$ such that $X, Y$ are compact, then
\[ s_j(AX - YB) \leq 2 \max(\|A\|, \|B\|)s_j(X \oplus Y) \tag{1.1} \]
for $j = 1, 2, \ldots$ Moreover, he generalized results of Bhatia-Kittaneh [2], Kittaneh [6], and Wang-Due [8]. In one of these generalizations he proved that if $A, B, X \in B(H)$ such that $A$ and $B$ are positive, and $X$ is compact, then
\[ s_j(AX - XB) \leq \max(\|A\|, \|B\|)s_j(X \oplus X) \tag{1.2} \]
for $j = 1, 2, \ldots$ If, in addition, $X$ is positive, Kittaneh has proved in [5] that
\[ s_j(AX - XA) \leq \frac{\|A\|}{2}s_j(X \oplus X) \tag{1.3} \]
for $j = 1, 2, \ldots$.

Hirzallah in [4] has proved a generalization to the inequality (1.1):
Let $A, B, X, Y \in B(H)$ such that $X$ and $Y$ are compact. Then
\[ s_j(AX - YB) \leq (\|A\| + \|B\|)s_j(X \oplus Y) \tag{1.4} \]
for $j = 1, 2, \ldots$.

It has been shown by Zhan in [9] that if $A, B \in K(H)$ are positive, then
\[ s_j(A - B) \leq s_j(A \oplus B) \tag{1.5} \]
for $j = 1, 2, \ldots$.

Kittaneh in [5] generalized the inequality (1.5) for generalized commutators:
If $A, B, X \in B(H)$ such that $A$ and $B$ are compact and positive, then
\[ s_j(AX - XB) \leq \|X\|s_j(A \oplus B) \tag{1.6} \]
for $j = 1, 2, \ldots$. 
Hirzallah in [4] generalized the inequality (1.6):

Let $A, B, X, Y$ be $n$-by-$n$ matrices such that $A$ and $B$ are positive semidefinite. Then

$$s_j(AX - YB) \leq \frac{\|X\| + \|Y\| + \|X - Y\|}{2} s_j(A \oplus B)$$

for $j = 1, 2, \ldots, n$. Moreover, Hirzallah has proved in [4]:

Let $A, B, X$ be $n$-by-$n$ matrices with polar decompositions $A = U|A|$, $B = V|B|$. Then

$$s_j(AX - XB) \leq \left(\frac{\|U\|_{X} - \|XV\|}{2}\right)s_j(A \oplus B)$$

for $j = 1, 2, \ldots, n$. In particular,

$$s_j(AX -XA) \leq \left(\frac{\|U\|_{X} - \|XU\|}{2}\right)s_j(A \oplus A)$$

for $j = 1, 2, \ldots, n$.

Our aim in this paper is to prove inequalities for singular values of $n$th-order Audeh generalized commutator, $A_1X_1 + \cdots + A_mX_m - Y_1B_1 - \cdots - Y_nB_n$ which will generalize the inequalities (1.1) to (1.9).

2. Main Results

We will present the major theorem which is an inequality for singular values of the $n$th order Audeh generalized commutator $A_1X_1 + \cdots + A_mX_m - Y_1B_1 - \cdots - Y_nB_n$. To prove this inequality, we need the following lemma, which is an immediate consequence of the min-max principle (see, e.g., [1, p.75] or [4, p.27]).

**Lemma 2.1.** Let $A, B, X \in B(H)$ such that $X$ is compact. Then

$$s_j(AXB) \leq \|A\| \|B\| s_j(X)$$

for $j = 1, 2, \ldots$. 

Our major theorem is a generalization of the inequality (1.4).

**Theorem 2.2.** Let $A_1, ..., A_m, B_1, ..., B_n, X_1, ..., X_m, Y_1, ..., Y_n \in B(H)$ such that $X_1, ..., X_m, Y_1, ..., Y_n$ are compact. Then

$$s_j (A_1 X_1 + \cdots + A_m X_m - Y_1 B_1 - \cdots - Y_n B_n) \leq (\sqrt{mM} + \sqrt{nN}) s_j (X_1 \oplus \cdots \oplus X_m \oplus Y_1 \oplus \cdots \oplus Y_n)$$

(2.2)

for $j = 1, 2, \ldots$, where

$$M = \sqrt{\|A_1\|^2 + \cdots + \|A_m\|^2} \quad \text{and} \quad N = \sqrt{\|B_1\|^2 + \cdots + \|B_n\|^2}.$$ 

**Proof.** Since

$$(A_1 X_1 + \cdots + A_m X_m - Y_1 B_1 - \cdots - Y_n B_n) \oplus 0$$

$$= \begin{bmatrix} A_1 & \cdots & A_m & I & \cdots & I \\ \vdots \\ I \\ B_1 \\ \vdots \\ B_n \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_m \\ -Y_1 \\ \vdots \\ -Y_n \end{bmatrix},$$

(2.3)

then

$$s_j (A_1 X_1 + \cdots + A_m X_m - Y_1 B_1 - \cdots - Y_n B_n)$$
More Commutator Inequalities for Hilbert Space Operators

\[
\begin{bmatrix}
A_1 & \ldots & A_m & I & \ldots & I
\end{bmatrix}
\begin{bmatrix}
X_1 \\
\vdots \\
X_m \\
- Y_1 \\
\vdots \\
- Y_n
\end{bmatrix}
= s_j
\begin{bmatrix}
I \\
\vdots \\
I \\
B_1 \\
\vdots \\
B_n
\end{bmatrix}
\begin{bmatrix}
I \\
\vdots \\
I \\
B_1 \\
\vdots \\
B_n
\end{bmatrix}
\begin{bmatrix}
I \\
\vdots \\
B_1 \\
\vdots \\
B_n
\end{bmatrix}
\]

(2.4)

\[
s_j(X_1 \oplus \cdots \oplus X_m \oplus Y_1 \oplus \cdots \oplus Y_n)
\leq \sqrt{||A_1^*||^2 + \cdots + ||A_m^*||^2 + nI ||B_1||^2 + \cdots + ||B_n||^2 + mI}\\
\leq s_j(X_1 \oplus \cdots \oplus X_m \oplus Y_1 \oplus \cdots \oplus Y_n)
\leq \sqrt{(||A_1||^2 + \cdots + ||A_m||^2 + n)(||B_1||^2 + \cdots + ||B_n||^2 + m)}\\
\leq s_j(X_1 \oplus \cdots \oplus X_m \oplus Y_1 \oplus \cdots \oplus Y_n)
\]

for \( j = 1, 2, \ldots \). In the inequality (2.4) replacing \( A_1, \ldots, A_m, B_1, \ldots, B_n \) by \( tA_1, \ldots, tA_m, tB_1, \ldots, tB_n \), for \( t > 0 \), respectively, we get
\[ s_j(A_1X_1 + \cdots + A_mX_m - Y_1B_1 - \cdots - Y_nB_n) \]
\[ \leq \sqrt{\left( t \left\| A_1 \right\|^2 + \cdots + t \left\| A_m \right\|^2 + n \right)(t \left\| B_1 \right\|^2 + \cdots + t \left\| B_n \right\|^2 + m)} \times s_j(X_1 \oplus \cdots \oplus X_m \oplus Y_1 \oplus \cdots \oplus Y_n) \] (2.5)

for \( j = 1, 2, \cdots \) and all \( t > 0 \). Since

\[
\min_{t>0} \sqrt{\left( t \left\| A_1 \right\|^2 + \cdots + t \left\| A_m \right\|^2 + n \right)(t \left\| B_1 \right\|^2 + \cdots + t \left\| B_n \right\|^2 + m)}
\] (2.6)

where \( M = \sqrt{\left\| A_1 \right\|^2 + \cdots + \left\| A_m \right\|^2} \) and \( N = \sqrt{\left\| B_1 \right\|^2 + \cdots + \left\| B_n \right\|^2} \). It follows from the inequalities (2.5), (2.6) that

\[ s_j(A_1X_1 + \cdots + A_mX_m - Y_1B_1 - \cdots - Y_nB_n) \leq (\sqrt{m}M + \sqrt{n}N)s_j(X_1 \oplus \cdots \oplus X_m \oplus Y_1 \oplus \cdots \oplus Y_n) \]
for \( j = 1, 2, \ldots \).

**Remark 1.** When replacing \( A_2 = \cdots = A_m = B_2 = \cdots = B_n = X_2 = \cdots = X_m = Y_2 = \cdots = Y_n = 0 \) in the inequality (2.2), we get the inequality (1.4).

Let \( A \in K(H) \) and let \( \alpha \) be complex number. The operator \( A - \alpha \) will be compact if \( H \) is \( r \)-dimensional Hilbert space or \( A \) is \( r \)-by-\( r \) matrix. So, our next results will be for \( r \)-by-\( r \) matrices (or operators on \( r \)-dimensional Hilbert space \( H \)).

To prove our next theorem, we need the following two lemmas.

**Lemma 2.3.** Let \( A \) be \( r \)-by-\( r \) positive semidefinite matrix and let

\[ \alpha_j = \frac{s_j(A)}{2} \text{ for } j = 1, 2, \ldots, r. \]
Then

\[ s_j(A - \alpha_j) = \alpha_j \] (2.7)

for \( j = 1, 2, \ldots, r. \)
Lemma 2.4. Let $A, B \in K(H)$. Then
\[
s_j(A + B) \leq s_j(A) + \|B\|
\] (2.8)
for $j = 1, 2, \ldots$

As an application of Theorem 2.2, we will present the following theorem which is a generalization of the inequalities (1.5), (1.6), and (1.7).

Theorem 2.5. Let $A_1, \ldots, A_m, B_1, \ldots, B_n, X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n$ be $r$-by-$r$ matrices such that $A_1, A_2, \ldots, A_m, B_1, \ldots, B_n$ are positive semidefinite, $M = \sqrt{\sum X_m^2 + \cdots + \sum X_m^2}$ and $N = \sqrt{\sum Y_1^2 + \cdots + \sum Y_n^2}$. Then
\[
s_j(A_1X_1 + \cdots + A_mX_m - Y_1B_1 - \cdots - Y_nB_n) \leq \frac{(\sqrt{mM} + \sqrt{nN}) + \|X_1 + \cdots + X_m - Y_1 - \cdots - Y_n\|}{2}
\times s_j(A_1 \oplus \cdots \oplus A_m \oplus B_1 \oplus \cdots \oplus B_n)
\] (2.9)
for $j = 1, 2, \ldots, r$. In particular,
\[
s_j(A_1X + \cdots + A_mX - XB_1 - \cdots - XB_n) \leq \frac{m + n + |m - n|}{2} \|X\| s_j(A_1 \oplus \cdots \oplus A_m \oplus B_1 \oplus \cdots \oplus B_n)
\] (2.10)
for $j = 1, 2, \ldots, r$.

Proof. It is well known that $s_j(T) = s_j(T^*)$ for $j = 1, 2, \ldots$. This implies that $s_j(A_1X_1 + \cdots + A_mX_m - Y_1B_1 - \cdots - Y_nB_n) = s_j(X_1^*A_1 + \cdots + X_m^*A_m - B_1Y_1^* - \cdots - B_nY_n^*)$ for $j = 1, 2, \ldots, r$.

By direct computations, we see that
\[
X_1^*A_1 + \cdots + X_m^*A_m - B_1Y_1^* - \cdots - B_nY_n^*
\]
\[ = X_1^* (A_1 - \gamma) + \cdots + X_m^* (A_m - \gamma) - (B_1 - \gamma) Y_1^* - \cdots - (B_n - \gamma) Y_n^* + \gamma (X_1^* + \cdots + X_m^* - Y_1^* - \cdots - Y_n^*) \]  
(2.11)

where \( \gamma \) is a complex number. Now, apply the inequality (2.8) we get

\[
s_j (X_1^* A_1 + \cdots + X_m^* A_m - B_1 Y_1^* - \cdots - B_n Y_n^*)
\]
\[
\leq s_j (X_1^* (A_1 - \gamma) + \cdots + X_m^* (A_m - \gamma) - (B_1 - \gamma) Y_1^* - \cdots - (B_n - \gamma) Y_n^*)
\]
\[
+ \| \gamma (X_1^* + \cdots + X_m^* - Y_1^* - \cdots - Y_n^*) \|
\]
(2.12)

for \( j = 1, 2, \ldots, r \). It follows from Theorem 2.2 that

\[
s_j (X_1^* A_1 + \cdots + X_m^* A_m - B_1 Y_1^* - \cdots - B_n Y_n^*)
\]
\[
\leq (\sqrt{mM} + \sqrt{nN}) s_j ((A_1 - \gamma) \oplus \cdots \oplus (A_m - \gamma) \oplus (B_1 - \gamma) \oplus \cdots \oplus (B_m - \gamma))
\]
\[
+ \| \gamma (X_1^* + \cdots + X_m^* - Y_1^* - \cdots - Y_n^*) \| \text{ for } j = 1, 2, \ldots, r,
\]
(2.13)

where \( M = \sqrt{\|X_1\|^2 + \cdots + \|X_m\|^2} \) and \( N = \sqrt{\|Y_1\|^2 + \cdots + \|Y_n\|^2} \).

Letting \( \gamma = \gamma_j = \frac{s_j (A_1 \oplus \cdots \oplus A_m \oplus B_1 \oplus \cdots \oplus B_n)}{2} \) for \( j = 1, 2, \ldots, r \),
we get

\[
s_j (X_1^* A_1 + \cdots + X_m^* A_m - B_1 Y_1^* - \cdots - B_n Y_n^*)
\]
\[
\leq (\sqrt{mM} + \sqrt{nN}) s_j ((A_1 - \gamma) \oplus \cdots \oplus (A_m - \gamma) \oplus (B_1 - \gamma) \oplus \cdots \oplus (B_m - \gamma))
\]
\[
+ \frac{s_j (A_1 \oplus \cdots \oplus A_m \oplus B_1 \oplus \cdots \oplus B_n)}{2} \| X_1^* + \cdots + X_m^* - Y_1^* - \cdots - Y_n^* \|
\]
(2.14)

Since \( A_1, \ldots, A_m, B_1, \ldots, B_n \) are positive semidefinite, it follows from Lemma 2.3 that
for \( j = 1, 2, \ldots, r \). Now, from the inequalities (2.14) and (2.15) we get

\[
s_j (A_1 X_1 + \cdots + A_m X_m - Y_1 B_1 - \cdots - Y_n B_n)
\leq \frac{\sqrt{mM} + \sqrt{nN}}{2} \| X_1 + \cdots + X_m - Y_1 \cdots - Y_n \|
\times s_j (A_1 \oplus \cdots \oplus A_m \oplus B_1 \oplus \cdots \oplus B_m)
\]

for \( j = 1, 2, \ldots, r \). Note that the inequality (2.9) is a generalization of the inequality (1.7). To see this, replace \( A_2 \cdots = A_m = B_2 = \cdots = B_n = X_2 = \cdots = X_m = Y_2 = \cdots = Y_n = 0 \), we get \( s_j (AX - YB) \leq \frac{\| X \| + \| Y \| + \| X - Y \|}{2} \times s_j (A \oplus B) \) for \( j = 1, 2, \ldots, r \).

As an application of Theorem 2.2, we will present the following theorem which is a generalization of the inequality (1.8).

**Theorem 2.6.** Let \( A_1, A_2, \ldots, A_n, X \) be \( r \times r \) matrices with polar decompositions \( A_1 = U_1 | A_1 |, A_2 = U_2 | A_2 |, \ldots, A_n = U_n | A_n | \). Then

\[
s_j (A_1 X - X A_2 - \cdots - X A_n)
\leq \frac{n \| X \| + \| U_1 X - X U_2 - \cdots - X U_n \|}{2}
\times s_j (A_1 \oplus A_2 \oplus \cdots \oplus A_n)
\]

for \( j = 1, 2, \ldots, r \).

**Proof.** As special case from Theorem 2.2, assume \( A_2 = \cdots = A_m = X_2 = \cdots = X_m = B_1 = Y_1 = 0, Y_i = X_i \) and \( B_i = A_i \) for \( i = 2, \ldots, n \), we get
\[ s_j(A_1 X_1 - X_2 A_2 - \cdots - X_n A_n) \]

\[ \leq (\| A_1 \| + \sqrt{n-1}L)s_j(X_1 \oplus X_2 \oplus \cdots \oplus X_n), \quad (2.17) \]

where

\[ L = \sqrt{\| A_2 \|^2 + \cdots + \| A_n \|^2}, \]

for \( j = 1, 2, \ldots \). Using the polar decomposition of \( A_1, A_2, \ldots, A_n \), and applying Theorem 2.5, we get

\[ s_j(A_1 X - X A_2 - \cdots - X A_n) \]

\[ = s_j(U_1 | A_1 | X - X U_2 | A_2 | - \cdots - X U_n | A_n |) \]

\[ = s_j(| A_1 | X - U_1^* X U_2 | A_2 | - \cdots - U_1^* X U_n | A_n |) \]

\[ \leq \left( \| X \| + \sqrt{(n-1)\| U_1^* X U_2 \|^2 + \cdots + \| U_1^* X U_n \|^2} \right) \]

\[ \times s_j(A_1 \oplus A_2 \oplus \cdots \oplus A_n) \]

\[ \leq \| X \| + \sqrt{(n-1)(n-1)\| X \|^2 \| U_1 X - X U_2 - \cdots - X U_n \|^2} \]

\[ \times s_j(A_1 \oplus A_2 \oplus \cdots \oplus A_n) \]

\[ \leq n\| X \| + \frac{\| U_1 X - X U_2 - \cdots - X U_n \|^2}{2} \]

\[ \times s_j(A_1 \oplus A_2 \oplus \cdots \oplus A_n) \quad (2.18) \]

for \( j = 1, 2, \ldots, r \).
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