Singular value inequalities for compact operators

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ABSTRACT

A singular value inequality due to Bhatia and Kittaneh says that if $A$ and $B$ are compact operators on a complex separable Hilbert space such that $A$ is self-adjoint, $B \geq 0$, and $\pm A \leq B$, then
$$s_j(A) \leq s_j(B \oplus B)$$
for $j = 1, 2, \ldots$. We give an equivalent inequality, which states that if $A$, $B$, and $C$ are compact operators such that
$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0,$$
then
$$s_j(B) \leq s_j(A \oplus C)$$
for $j = 1, 2, \ldots$. Moreover, we give a sharper inequality and we prove that this inequality is equivalent to three equivalent inequalities considered by Tao. In particular, we show that if $A$ and $B$ are compact operators such that $A$ is self-adjoint, $B \geq 0$, and $\pm A \leq B$, then
$$2s_j(A) \leq s_j((B + A) \oplus (B - A))$$
for $j = 1, 2, \ldots$. Some applications of these results will be given.

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1. Introduction

Let $B(H)$ denote the space of all bounded linear operators on a complex separable Hilbert space $H$, and let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. For $T \in K(H)$, the singular values of $T$, denoted by $s_1(T)$, $s_2(T)$, ... are the eigenvalues of the positive operator $|T| = (T^*T)^{1/2}$ enumerated as $s_1(T) \geq s_2(T) \geq \ldots$ and repeated according to multiplicity. Note that $s_j(T) = s_j(T^*) = s_j(|T|)$ for $j = 1, 2, \ldots$. It follows by Weyl's monotonicity principle (see, e.g., [1, p. 63] or [4, p. 26]) that if $S, T \in K(H)$ are positive and $S \leq T$, then $s_j(S) \leq s_j(T)$ for $j = 1, 2, \ldots$. Moreover, for $S, T \in K(H)$,
\[ s_j(S) \leq s_j(T) \text{ if and only if } s_j(S \oplus S) \leq s_j(T \oplus T) \text{ for } j = 1, 2, \ldots \] The singular values of \( S \oplus T \) and 
\[
\begin{bmatrix}
0 & T \\
S & 0
\end{bmatrix}
\]
are the same, and they consist of those of \( S \) together with those of \( T \). Here, we use the direct sum notation \( S \oplus T \) for the block-diagonal operator 
\[
\begin{bmatrix}
S & 0 \\
0 & T
\end{bmatrix}
\]
defined on \( H \oplus H \).

Bhatia and Kittaneh have proved in [3] that if \( A, B \in K(H) \) such that \( A \) is self-adjoint, \( B \geq 0 \), and \( \pm A \leq B \), then
\[
s_j(A) \leq s_j(B \oplus B) \tag{1.1}
\]
for \( j = 1, 2, \ldots \)
We will give a new equivalent form of (1.1):

If \( A, B, C \in K(H) \) such that 
\[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \succeq 0,
\]
then
\[
s_j(B) \leq s_j(A \oplus C) \tag{1.2}
\]
for \( j = 1, 2, \ldots \)

The well-known arithmetic-geometric mean inequality for singular values, due to Bhatia and Kittaneh [2], says that if \( A, B \in K(H) \), then
\[
2s_j(AB^*) \leq s_j(A^*A + B^*B) \tag{1.3}
\]
for \( j = 1, 2, \ldots \). On the other hand, Zhan has proved in [8] that if \( A, B \in K(H) \) are positive, then
\[
s_j(A - B) \leq s_j(A \oplus B) \tag{1.4}
\]
for \( j = 1, 2, \ldots \). Moreover, Tao has proved in [7] that if \( A, B, C \in K(H) \) such that 
\[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \succeq 0,
\]
then
\[
2s_j(B) \leq s_j \begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \tag{1.5}
\]
for \( j = 1, 2, \ldots \)

It has been pointed out in [7] that the three inequalities (1.3)-(1.5) are equivalent. It should be mentioned here that while the inequalities in [3,7,8] are formulated for matrices, they can be extended in a natural way to compact operators on a complex separable Hilbert space. In this paper, we will give a new inequality, which is equivalent to these inequalities:

If \( A, B \in K(H) \) such that \( A \) is self-adjoint, \( B \geq 0 \), and \( \pm A \leq B \), then
\[
2s_j(A) \leq s_j((B + A) \oplus (B - A)) \tag{1.6}
\]
for \( j = 1, 2, \ldots \). This inequality is sharper than (1.1).

2. Main results

Our first singular value inequality is equivalent to the inequality (1.1).

**Theorem 2.1.** Let \( A, B, C \in K(H) \) such that 
\[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \succeq 0.
\]
Then
\[
s_j(B) \leq s_j(A \oplus C)
\]
for \( j = 1, 2, \ldots \)
Proof. Since \[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \succeq 0,
\] it follows that \[
\begin{bmatrix}
A & -B \\
-B^* & C
\end{bmatrix} \succeq 0.\] In fact, if \( U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \), then \( U \) is unitary and
\[
\begin{bmatrix}
A & -B \\
-B^* & C
\end{bmatrix} = U \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} U^* \succeq 0.\] Thus, \[
\begin{bmatrix}
A & 0 \\
0 & C
\end{bmatrix} \succeq \pm \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix},\] and so by applying the inequality (1.1), we get
\[
s_j(B \oplus B^*) \leq s_j((A \oplus C) \oplus (A \oplus C)) \quad \text{for } j = 1, 2, \ldots.\] This is equivalent to saying that \( s_j(B) \leq s_j(A \oplus C) \) for \( j = 1, 2, \ldots. \) □

Remark 2.2. While the proof of the inequality (1.2), given in Theorem 2.1 is based on the inequality (1.1), it can be obtained by employing the inequality (1.5) as follows:

If \[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \succeq 0,
\] then\[
\begin{bmatrix}
A & -B \\
-B^* & C
\end{bmatrix} \succeq 0,\] and so
\[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \preceq \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} + \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix} = 2 \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \succeq 0.\] By Weyl’s monotonicity principle, we have
\[
s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) \leq 2s_j\left(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}\right) = 2s_j(A \oplus C) \quad \text{for } j = 1, 2, \ldots.\] Chaining this with the inequality (1.5), yields the inequality (1.2).

Now, we prove that the inequalities (1.1) and (1.2) are equivalent.

Theorem 2.3. The following statements are equivalent:

(i) Let \( A, B \in \mathcal{K}(H) \), where \( A \) is self-adjoint, \( B \succeq 0, \) and \( \pm A \leq B. \) Then
\[
s_j(A) \leq s_j(B \oplus B)\]
for \( j = 1, 2, \ldots. \)

(ii) Let \( A, B, C \in \mathcal{K}(H) \) such that
\[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \succeq 0.\] Then
\[
s_j(B) \leq s_j(A \oplus C)\]
for \( j = 1, 2, \ldots. \)

Proof.

(i) \( \Rightarrow \) (ii) This implication follows from the proof of Theorem 2.1.

(ii) \( \Rightarrow \) (i) Let \( A, B \in \mathcal{K}(H) \), where \( A \) is self-adjoint, \( B \succeq 0, \) and \( \pm A \leq B. \) Then the matrix \[
\begin{bmatrix}
B & A \\
A & B
\end{bmatrix} \succeq 0.\] In fact, if \( U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \), then \( U \) is unitary and
\[
\begin{bmatrix}
B & A \\
A & B
\end{bmatrix} = U^* \begin{bmatrix} B + A & 0 \\ 0 & B - A \end{bmatrix} U \succeq 0.\] Thus, by (ii) we have \( s_j(A) \leq s_j(B \oplus B) \) for \( j = 1, 2, \ldots. \) □

Our second singular value inequality is equivalent to the inequalities (1.3)-(1.5).
Theorem 2.4. Let \( A, B \in K(H) \), where \( A \) is self-adjoint, \( B \geq 0 \), and \( \pm A \leq B \). Then
\[
2 s_j(A) \leq s_j((B + A) \oplus (B - A))
\]
for \( j = 1, 2, \ldots \)

Proof. Let \( X = \begin{bmatrix} B & A \\ A & B \end{bmatrix} \). Since \( \begin{bmatrix} B & A \\ A & B \end{bmatrix} \) is unitarily equivalent to \( \begin{bmatrix} B + A & 0 \\ 0 & B - A \end{bmatrix} \), and since \( \pm A \leq B \), it follows that \( X \) is positive. Now, applying the inequality (1.5) to the operator \( X \), we get
\[
2 s_j(A) \leq s_j(B + A) = s_j(B + A \oplus (B - A)) \text{ for } j = 1, 2, \ldots \quad \square
\]

Remark 2.5. While the proof of the inequality (1.6), given in Theorem 2.4 is based on the inequality (1.5), it can be obtained by applying the inequality (1.4) to the positive operators \( B + A \) and \( B - A \).

The inequality (1.6) is sharper than (1.1). In fact, if \( A, B \in K(H) \), where \( A \) is self-adjoint, \( B \geq 0 \), and \( \pm A \leq B \), then by Weyl’s monotonicity principle, we have
\[
s_j((B + A) \oplus (B - A)) = s_j(B + A) \leq s_j(B + 2B) = 2 s_j(B \oplus B)
\]
for \( j = 1, 2, \ldots \)

Theorem 2.6. The following statements are equivalent:

(i) Let \( A, B \in K(H) \) be positive operators. Then
\[
s_j(A - B) \leq s_j(A \oplus B)
\]
for \( j = 1, 2, \ldots \)

(ii) Let \( A, B \in K(H) \). Then
\[
2 s_j(AB^*) \leq s_j(A^*A + B^*B)
\]
for \( j = 1, 2, \ldots \)

(iii) Let \( A, B, C \in K(H) \) such that \( \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \) \( \geq 0 \). Then
\[
2 s_j(B) \leq s_j(A \oplus B)
\]
for \( j = 1, 2, \ldots \)

(iv) Let \( A, B \in K(H) \) such that \( A \) is self-adjoint, \( B \geq 0 \), and \( \pm A \leq B \). Then
\[
2 s_j(A) \leq s_j((B + A) \oplus (B - A))
\]
for \( j = 1, 2, \ldots \)
Proof. Note that (i), (ii), and (iii) are equivalent by [7]. We will prove that (iii) is equivalent to (iv), and this will complete the proof of the theorem.

(iii) ⇒ (iv) This implication follows from the proof of Theorem 2.4.

(iv) ⇒ (iii) Assume that \[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \geq 0.
\] Since \[
\begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix}
\] and \[
\begin{bmatrix}
A & -B \\
-B^* & C
\end{bmatrix}
\] are unitarily equivalent, it follows that \[
\begin{bmatrix}
A & -B \\
-B^* & C
\end{bmatrix} \geq 0, \text{ and so } \begin{bmatrix}
A & 0 \\
0 & C
\end{bmatrix} \geq \pm \begin{bmatrix}
0 & B \\
B^* & 0
\end{bmatrix}. \]
Now, applying (iv), we get \[
2s_j(B \oplus B^*) \leq s_j(B) \leq s_j(B \oplus B^*)\]
for \(j = 1, 2, \ldots\) This is equivalent to saying that \[
s_j(B) = s_j(B \oplus B^*)\]
for \(j = 1, 2, \ldots\)
\(\Box\)

As an application of our results, we present the following theorem, which is sharper and more general than the Bhatia–Kittaneh inequality given in [3], which states that if \(A, B \in K(H)\), then
\[
s_j(AB^* + BA^*) \leq s_j((AA^* + BB^*) \oplus (AA^* + BB^*))
\]
(2.1)
for \(j = 1, 2, \ldots\). For related Cauchy–Schwarz type inequalities, we refer to [5] and references therein.

**Theorem 2.7.** Let \(A, B \in K(H)\), and let \(X \in B(H)\). Then
\[
2s_j(AX^*B^* + BXA^*)
\]
\[
\leq s_j((A|X|A^* + B|X^*|B^* + AX^*B^* + BXA^*)
\]
\[
\oplus (A|X|A^* + B|X^*|B^* - AX^*B^* - BXA^*)
\]
(2.2)
for \(j = 1, 2, \ldots\)

**Proof.** Let \(Y = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\) and \(Z = \begin{bmatrix} |X| & \pm X^* \\ \pm X & |X^*| \end{bmatrix}\). Then \(Z \geq 0\) (see, e.g., [6]), and
\[
YZY^* = \begin{bmatrix}
A|X|A^* \pm BXA^* \pm AX^*B^* + B|X^*|B^* & 0 \\
0 & 0
\end{bmatrix}
\]
\[
\geq 0,
\]
which implies that
\[
A|X|A^* + B|X^*|B^* \geq \pm (AX^*B^* + BXA^*).
\]
(2.3)
Now, applying Theorem 2.4 to the operators \(AX^*B^* + BXA^*\) and \(A|X|A^* + B|X^*|B^*\) we get the result. \(\Box\)

**Remark 2.8.** Letting \(X = I\) in Theorem 2.7, we have
\[
2s_j(AB^* + BA^*)
\]
\[
\leq s_j((AA^* + BB^* + AB^* + BA^*) \oplus (AA^* + BB^* - AB^* - BA^*))
\]
(2.4)
for \(j = 1, 2, \ldots\).

In view of Weyl’s monotonicity principle and the fact that \(\pm (AB^* + BA^*) \leq AA^* + BB^*\), one can see that the inequality (2.4) is sharper than the inequality (2.1).
If $A, B \in B(H)$, then the operator matrix
\[
\begin{bmatrix}
|A| + |B| & A + B \\
A^* + B^* & |A^*| + |B^*| 
\end{bmatrix}
\]
is positive, and so if $A, B \in K(H)$, then it follows from the inequalities (1.2) and (1.5), respectively, that
\[
s_j(A + B) \leq s_j((|A| + |B|) \oplus (|A^*| + |B^*|))
\]  
for $j = 1, 2, \ldots$ and
\[
2s_j(A + B) \leq s_j\left[
\begin{bmatrix}
|A| + |B| & A + B \\
A^* + B^* & |A^*| + |B^*| 
\end{bmatrix}
\right]
\]
for $j = 1, 2, \ldots$

The inequality (2.5), which is a triangle-type inequality has been obtained earlier in [3].

Our final application is a singular value inequality involving the Jordan parts of compact self-adjoint operators. The Jordan decomposition asserts that every self-adjoint operator can be expressed as the difference of two positive operators. In fact, if $A \in B(H)$ is self-adjoint, then $A = A^+ - A^-$, where $A^+$ and $A^-$ are the positive operators given by $A^+ = \frac{|A| + A}{2}$ and $A^- = \frac{|A| - A}{2}$.

**Theorem 2.9.** Let $A, B \in K(H)$ be self-adjoint operators. Then
\[
s_j(A + B) \leq s_j((A^+ + B^+) \oplus (A^- + B^-))
\]  
for $j = 1, 2, \ldots$

**Proof.** Since $A$ and $B$ are self-adjoint, it follows that $\pm A \leq |A|$ and $\pm B \leq |B|$, and so $\pm (A + B) \leq |A| + |B|$. Now, applying Theorem 2.4, we have
\[
2s_j(A + B) \leq s_j((|A| + |B| + A + B) \oplus (|A| + |B| - (A + B))
\]
\[
= s_j((|A| + A + |B| + B) \oplus (|A| - A + |B| - B))
\]
\[
= s_j((2A^+ + 2B^+) \oplus (2A^- + 2B^-))
\]
\[
= 2s_j((A^+ + B^+) \oplus (A^- + B^-))
\]
for $j = 1, 2, \ldots$, and so $s_j(A + B) \leq s_j((A^+ + B^+) \oplus (A^- + B^-))$ for $j = 1, 2, \ldots$ \(\square\)

It can be easily verified that the inequality (2.7) is a generalization of the inequality (1.4). In fact, if $A, B \in K(H)$ are positive, then by the inequality (2.7), we have
\[
s_j(A - B) = s_j(A + (-B))
\]
\[
\leq s_j((A^+ + (-B)^+) \oplus (A^- + (-B)^-))
\]
\[
= s_j((A + 0) \oplus (0 + B))
\]
\[
= s_j(A \oplus B)
\]
for $j = 1, 2, \ldots$

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**References**